9th lecture

FERMION FIELDS

after the toolbox, now the real stuff...

so far discussed Hamiltonians & Lagrangians
for bosonic fields

but Universe contains also fermions (leptons & quarks)

these are very different quanta — bosons socialize / fermions avoid each other

different statistics:

$$\psi(\gamma_1, \gamma_2, \dots) = +\psi(\gamma_2, \gamma_1, \dots)$$

$$\psi(e_1, e_2, \dots) = -\psi(e_2, e_1, \dots)$$

in nonrelativistic quantum mechanics, an addition postulate
its origin remains obscure — it comes from relativistic QFT

Pauli - Lüders: Spin - Statistics Theorem

First try

- seek a relativistic Lagrangian describing spin-$\frac{1}{2}$ fields

recall: $SO(1,3) \simeq SU(2)_L \times SU(2)_R$ reps labelled by (j_L, j_R)

basic spinor reps.: $(\frac{1}{2}, 0)$ left-handed Weyl spinor $\xi_{\alpha=1,2}$

$(0, \frac{1}{2})$ right-handed Weyl spinor $\eta_{\dot\alpha=1,2}$

start with $\xi_\alpha(\vec r, t) \in (\frac{1}{2}, 0) \rightsquigarrow (\xi_\alpha)^* = \xi^*_{\dot\alpha} \in (0, \frac{1}{2})$

$\left[\sigma_\mu = (\mathbb{1}, \vec\sigma) \right.$, $\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta$, $\xi^\alpha \Box \xi_\alpha$ or $\partial_\mu \xi^\alpha \partial^\mu \xi_\alpha$ $\left. \right]$

are not real !

next best:

$$\mathcal{L}^{\text{fermi}}_{\text{kin}} = i\, \xi^\alpha (\sigma^\mu)_{\alpha\dot\beta}\, \partial_\mu\, \xi^{*\dot\beta} = i\, \xi^\alpha\, \partial_{\alpha\dot\beta}\, \xi^{*\dot\beta}$$

for hermiticity

by construction Lorentz-invariant & hermitian

describes massless lefthanded fermion & its antiparticle

dimensions: $[\mathcal{L}] = M^4 \rightsquigarrow [\xi] = M^{3/2} \Leftrightarrow [\phi] = [A] = M$

- analogous to bosonic quantum field:

  put system into a box $L \times L \times L$ $(V = L^3)$

  $\xi_\alpha(x+L, y, z, t) = \xi_\alpha(x, y+L, z, t) = \xi_\alpha(x, y, z+L, t) = \xi_\alpha(x, y, z, t)$

  periodic boundary conditions

  Fourier expansion: $\xi_\alpha(\vec{r}, t) = \sum_{\vec{n} \in \mathbb{Z}^3} \xi_\alpha^{\vec{n}}(t)\, e^{2\pi i \vec{n}\cdot\vec{r}/L}$

  $$L = \int d^3x\, \mathcal{L} = V \sum_{\vec{n}} \left\{ i\, \xi^\alpha_{\vec{n}}\, \underset{\uparrow\ \sigma^0 = \mathbb{1}}{\delta_{\alpha\dot\beta}}\, \frac{d}{dt} \xi^{*\dot\beta}_{\vec{n}} + \frac{2\pi n_i}{L}\, \xi^\alpha_{\vec{n}}\, \underset{\uparrow\ \sigma^{\mu=i} = -\sigma_i}{(\sigma_i)_{\alpha\dot\beta}}\, \xi^{*\dot\beta}_{\vec{n}} \right\}$$

  $\rightsquigarrow$ sum of non-interacting complex modes $\xi^\alpha_{\vec{n}}(t)$, 1^{st} order in ∂_t

- focus on one of these modes, say $\vec{n} = (0,0,1)$, put $L = 1$

  $\rightsquigarrow L_{(001)} = i\left(\xi^1 \dot\xi^{*1} + \xi^2 \dot\xi^{*2}\right) + 2\pi\left(\xi^1 \xi^{*1} - \xi^2 \xi^{*2}\right)$ $\Big|\ \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

  canonical momenta: $\Pi_{\xi^{*\dot\beta}} = i\,\xi^\beta$, $\Pi_{\xi^\alpha} = -i\,\xi^{*\dot\alpha}$ | constraint: use only <u>one</u>

  canonical pairs $(\xi^\alpha, \xi^{*\dot\alpha})$ $\rightsquigarrow$ Poisson bracket $\{\xi^\alpha, \xi^{*\dot\alpha}\} = -i$

$$H_{(001)} = \Pi_{\xi^{*\beta}} \dot{\xi}^{*\beta} - L_{(001)} = 2\pi\left(\xi^2 \xi^{*2} - \xi^1 \xi^{*1}\right)$$

[normal for 1^{st}-order Lagrangians $L = q\dot{q} \rightsquigarrow \pi = q \rightsquigarrow \pi\dot{q} = q\dot{q} \rightsquigarrow H = V$]

- this is a pair of harmonic oscillators!

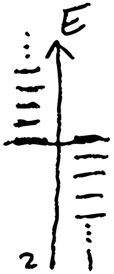
rename $\xi^1 \to b$, $\xi^2 \to a$, $\omega = \frac{2\pi}{L}$

$$H_{osc} = \omega\left(aa^* - bb^*\right) \quad \text{with } \{a, a^*\} = \{b, b^*\} = -i$$

quantize: $a \to \hat{a}$, $a^* \to \hat{a}^\dagger$, $b \to \hat{b}$, $b^* \to \hat{b}^\dagger$, $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$

symmetrize $\rightsquigarrow$ $\hat{H}_{osc} = \frac{\omega}{2}\left(aa^\dagger + a^\dagger a - bb^\dagger - b^\dagger b\right) = \omega\left(a^\dagger a - b^\dagger b\right)$

- energy spectrum $a|0\rangle = b|0\rangle = 0$



$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1! n_2!}} (b^\dagger)^{n_1} (a^\dagger)^{n_2} |0\rangle$$

$$\hat{H}_{osc}|n_1, n_2\rangle = E_{n_1,n_2}|n_1, n_2\rangle \text{ with } E_{n_1,n_2} = \omega(n_2 - n_1) = E_{(001)} \to ☹$$

$|n_1, n_2\rangle$ = state with n_1 particles +ve & n_2 particles with -ve helicity

each with momentum $\vec{p} = \frac{2\pi}{L}\begin{pmatrix}0\\0\\1\end{pmatrix}$ | $\frac{1}{2}\vec{\sigma}\cdot\vec{n}\begin{pmatrix}1\\0\end{pmatrix} = +\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}$, $\frac{1}{2}\vec{\sigma}\cdot\vec{n}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}$

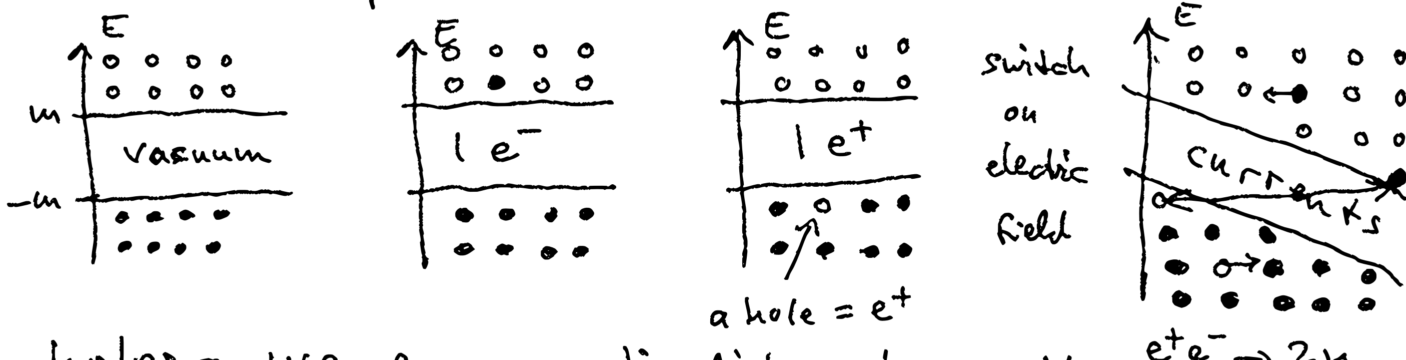
$\longrightarrow$ no ground state ↯

the Dirac sea

Dirac's proposal to solve the no-ground-state problem:

- electron has mass $m \rightsquigarrow E_{\vec{p}} = \pm\sqrt{\vec{p}^2+m^2}$, mass gap (no ext. field)
- accept Pauli's principle: each quantum state occupiable once
- new vacuum ("Dirac sea"): all states with $E_{\vec{p}} < 0$ filled

$$|sea\rangle = \prod_{\vec{p}} b^{+}_{\vec{p}}\,|0\rangle$$ & subtract infinite energy shift



holes = +ve-energy antiparticles solves problem

but: how to derive Pauli principle from Dirac equation?

answer: Pauli-Fierz (1940) , Berezin (~1960s)

fermions need ANTIcommutation relations — classical fermi fields are Grassmann

excursion: Grassmann algebra

"anticommuting numbers"

- let $\{a_i\}_{i=1,\dots,n}$ be a basis with $a_i a_j + a_j a_i = 0$ $\rightarrow a_i^2 = 0$

  element of Grassmann algebra is a function

  $$f(a) = c_0 + c_i a_i + c_{ij} a_i a_j + c_{ijk} a_i a_j a_k + \dots$$

  where $c_0, c_i, c_{ij}, \dots \in \mathbb{R}$ or $\mathbb{C}$. Series terminates at power a^{n+1}

  even powers of a_i are "even" $\rightsquigarrow$ commute with all

  odd powers of a_i are "odd" $\rightsquigarrow$ anticommute with odd ones

  $\rightsquigarrow$ only c_0 is a regular number, other even parts are nilpotent
- addition & multiplication & scalar multiplication $\rightarrow$ algebra
- differentiation: $\frac{\partial}{\partial a_i} 1 := 0$, $\frac{\partial}{\partial a_i} a_j := \delta_{ij}$

  graded Leibniz rule: $\frac{\partial}{\partial a_2}(a_1 a_2) = \left(\frac{\partial}{\partial a_2} a_1\right) a_2 - a_1 \left(\frac{\partial}{\partial a_2} a_2\right) = -a_1$
- integration: $\int da_i\, f(a) := \partial f / \partial a_i$ just a linear functional
- conjugation: $n = 2m$ even $\rightsquigarrow$ divide $\{a_1, \dots, a_{2m}\} = \{a_1, \dots, a_n, a_1^\dagger, \dots a_n^\dagger\}$

  define involution $a_i \leftrightarrow a_i^\dagger$ with $c_{\dots} \leftrightarrow c_{\dots}^*$ $\rightsquigarrow$ $f(a) \leftrightarrow f^\dagger(a)$, $(fg)^\dagger = g^\dagger f^\dagger$

Grassmann waves

- switch classical fermion variables to Grassmann numbers

$$L = -i\dot{\xi}\xi^\dagger + \omega\xi\xi^\dagger = i\xi^\dagger\dot{\xi} - \omega\xi^\dagger\xi$$

can also move ∂_t onto $\xi^\dagger$ via partial integration

with $\xi\xi^\dagger + \xi^\dagger\xi = 0$, $\xi^2 = 0 = \xi^{\dagger 2}$, $\{\xi, \xi^\dagger\} = -i$

e.o.m.: $-\frac{d}{dt}\frac{\partial L}{\partial \dot{\xi}} + \frac{\partial L}{\partial \xi} = 0 \rightsquigarrow i\dot{\xi}^\dagger + \omega\xi^\dagger = 0$

hermition conjugate $\rightsquigarrow i\dot{\xi} - \omega\xi = 0$

note: $L^\dagger = L$ because $(\dot{\xi}\xi^\dagger)^\dagger = \xi\dot{\xi}^\dagger \xrightarrow{p.i.} -\dot{\xi}\xi^\dagger$

$(\xi\xi^\dagger)^\dagger = \xi\xi^\dagger$

Poisson bracket:

$$\{f, g\} := -i\left(\frac{\partial f}{\partial \xi}\frac{\partial g}{\partial \xi^\dagger} + \frac{\partial f}{\partial \xi^\dagger}\frac{\partial g}{\partial \xi}\right) \quad \text{order relevant}$$

time evolution in Grassmann phase space spanned by ξ & $\xi^\dagger$

$$\frac{df}{dt} = \{H, f\} \text{ with } H = \omega\xi^\dagger\xi \rightsquigarrow \left\{\begin{array}{l} \dot{\xi} = \omega\{\xi^\dagger\xi, \xi\} = -i\omega\xi \\ \dot{\xi}^\dagger = \omega\{\xi^\dagger\xi, \xi^\dagger\} = i\omega\xi^\dagger \end{array}\right\} \checkmark$$

- quantization for phase space $\{q_i, p_i, \xi_\alpha, \xi^+_\alpha\}$

for f <u>or</u> g even: $i\{f,g\} \longrightarrow [\hat f, \hat g] \equiv \hat f \hat g - \hat g \hat f$ commutator

if f <u>and</u> g odd: $i\{f,g\} \longrightarrow [\hat f, \hat g]_+ = \hat f \hat g + \hat g \hat f$ anticommutator

e.g. $[\hat\xi_\alpha, \hat\xi^+_\beta]_+ = \delta_{\alpha\beta}$, $[\hat\xi_\alpha, \hat\xi_\beta]_+ = 0 = [\hat\xi^+_{\dot\alpha}, \hat\xi^+_{\dot\beta}]_+$

realize in "ξ space": $\hat\xi \to \xi$, $\hat\xi^+ \to \partial/\partial\xi$

[check: $[\hat\xi, \hat\xi^+]_+ f = \xi \frac{\partial}{\partial\xi} f + \frac{\partial}{\partial\xi}(\xi f) = f$ ✓]

$$\Pi_\xi = \frac{\partial L}{\partial\dot\xi} = -i\xi^+ \longrightarrow -i\frac{\partial}{\partial\xi}$$

$$H = \omega\xi^+\xi = \frac{\omega}{2}(\xi^+\xi - \xi\xi^+) \longrightarrow \hat H = \frac{\omega}{2}\left(\frac{\partial}{\partial\xi}\xi - \xi\frac{\partial}{\partial\xi}\right)$$

- spectrum: wave functions $\psi(\xi) = a + b\xi$ linear!

Hilbert space is 2-dim'l

energy eigenstates: $\psi(\xi) = 1$ $\quad E = \frac{\omega}{2}$

$\psi(\xi) = \xi$ $\quad E = -\frac{\omega}{2}$

$\Big\}$ Pauli principle emerges!

"fermionic harmonic oscillator"

- quantization

for bosons: commutative algebra $\xrightarrow{qu}$ Heisenberg algebra $[\hat{q},\hat{p}]=i$

for fermions: Grassmann algebra $\xrightarrow{qu}$ Clifford algebra $[\hat{\xi},\hat{\xi}^{+}]_{+}=1$

- back to $L_{(001)} = i\left(\xi^1\dot{\xi}^{+1} + \xi^2\dot{\xi}^{+2}\right) + \frac{2\pi}{L}\left(\xi^1\xi^{+1} - \xi^2\xi^{+2}\right)$

now assume ξ^α, $\xi^{+\beta}$ to be Grassmannian

$$H = \frac{2\pi}{L}\left(-\xi^1\xi^{+1} + \xi^2\xi^{+2}\right)$$

$$\overset{\text{antisym.}}{=} \frac{\pi}{L}\left(\xi^{+1}\xi^1 - \xi^1\xi^{+1} - \xi^{+2}\xi^2 + \xi^2\xi^{+2}\right)$$

quantize $\rightsquigarrow$

$$\hat{H} = \frac{\pi}{L}\left(\frac{\partial}{\partial\xi^1}\xi^1 - \xi^1\frac{\partial}{\partial\xi^1} - \frac{\partial}{\partial\xi^2}\xi^2 + \xi^2\frac{\partial}{\partial\xi^2}\right)$$

- spectrum: wave functions $\psi(\xi^1,\xi^2) = a + b\xi^1 + c\xi^2 + d\xi^1\xi^2$

Hilbert space is 4-dim'l, energy eigenbasis:

$$\left.\begin{array}{lll} |10\rangle & \psi = \xi^1 & E = -\frac{2\pi}{L} \\ |00\rangle & \psi = 1 & E = 0 \\ |11\rangle & \psi = \xi^1\xi^2 & E = 0 \\ |01\rangle & \psi = \xi^2 & E = +\frac{2\pi}{L} \end{array}\right\}$$

bounded from below
no infinite towers
ξ^1 carries +ve helicity
ξ^2 carries −ve helicity

- full Hamiltonian is a sum over $\vec{n} \in \mathbb{Z}^3$ $\to$

eigenstates = tensor products $\bigotimes_{\vec{n}} |..\rangle_{\vec{n}}$ {Dirac sea interpretation}

$\|10\rangle_{\vec{n}}$	$\xi^1_{\vec{n}}$	$E_{\vec{n}} = -\frac{2\pi}{L}\|\vec{n}\|$	$\longrightarrow$ Dirac see vacuum (neg. state occupied)
$\|00\rangle_{\vec{n}}$	1	$E_{\vec{n}} = 0$	$\longrightarrow$ right-handed hole, $\vec{p} = -\frac{2\pi\vec{n}}{L}$
$\|11\rangle_{\vec{n}}$	$\xi^1_{\vec{n}}\xi^2_{\vec{n}}$	$E_{\vec{n}} = 0$	$\longrightarrow$ left-handed particle, $\vec{p} = +\frac{2\pi\vec{n}}{L}$
$\|01\rangle_{\vec{n}}$	$\xi^2_{\vec{n}}$	$E_{\vec{n}} = +\frac{2\pi}{L}\|\vec{n}\|$	$\longrightarrow$ both particle + hole

change the zero-energy (and momentum) level such that

$|10\rangle_{\vec{n}} =: |vac\rangle_{\vec{n}}$ hat $E'_{\vec{n}} = 0$ (Dirac sea vacuum)

quartet of states:

	$E'_{\vec{n}}$	$\vec{p}'_n$
- vacuum	0	0
- left-handed particle e^-_L	$\frac{2\pi}{L}\|\vec{n}\|$	$\frac{2\pi}{L}\vec{n}$
- right-handed antiparticle e^+_R	$\frac{2\pi}{L}\|\vec{n}\|$	$-\frac{2\pi}{L}\vec{n}$
- both	$\frac{4\pi}{L}\|\vec{n}\|$	0

$\rightsquigarrow \mathcal{L}^{fermi}_{kin} = i\xi^\alpha (\sigma^\mu)_{\alpha\dot{\beta}}\, \xi^{+\dot{\beta}}$ describes e^-_L & e^+_R (with mass $= 0$)

electromagnetic interaction: Dirac equation

for a description of electrons must couple to A_μ

- try to write Lorentz- & gauge-invariant $\mathcal{L}$ for ξ^α & A_μ with minimal interaction:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\, \xi^\alpha (\sigma^\mu)_{\alpha\dot\beta} (\partial_\mu - ie A_\mu) \xi^{\dagger\dot\beta}$$

is invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \,, \quad \xi_\alpha(x) \to e^{-ie\lambda(x)} \xi_\alpha(x)$$

covariant derivative

$$D_\mu = \partial_\mu + ie A_\mu \qquad (D_\mu \xi^\alpha ,\ D^\dagger_\mu \xi^{\dagger\dot\beta} \text{ transform like } \xi^\alpha,\ \xi^{\dagger\dot\beta})$$

- not full realistic because
  - theoretically: has a "chiral anomaly" at qu. level $\to$ $\left\{ \begin{array}{l} \text{non-} \\ \text{renorma-} \\ \text{lizable} \end{array} \right\}$
  - phenomenologically: need e^-_R & e^+_L

$\rightsquigarrow$ double degrees of freedom by adding a second copy with another fermi field $\eta_\alpha(x)$ of opposite charge: $\eta \to e^{+ie\lambda}\eta$

- $\mathcal{L}_{\xi\eta} = i\xi^\alpha (\sigma^\mu)_{\alpha\dot\beta}(\partial_\mu - ieA_\mu)\xi^{\dagger\dot\beta} + i\eta^\alpha(\sigma^\mu)_{\alpha\dot\beta}(\partial_\mu + ieA_\mu)\eta^{\dagger\dot\beta}$

quantization yields: $\xi_\alpha \to e_L^-, e_R^+$, $\eta_\alpha \to e_L^+, e_R^-$

now, also a mass term is possible

$$\mathcal{L}_m = m\left(\xi_\alpha \eta^\alpha + \eta^{\dagger\dot\alpha}\xi^\dagger_{\dot\alpha}\right)$$ Dirac mass term

for a single fermion it is possible to write

$$\mathcal{L}_{Maj} = m\left(\xi_\alpha \xi^\alpha + \xi^{\dagger\dot\alpha}\xi^\dagger_{\dot\alpha}\right)$$ Majorana mass term

but only for electrically <u>neutral</u> fermions
(since it is not gauge-invariant for charged fermions)

full QED: $\mathcal{L}_{QED} = \mathcal{L}_{Maxwell} + \mathcal{L}_{\xi\eta} + \mathcal{L}_m$

- more compact notation: Dirac (bi)spinor

$$\Psi = \begin{pmatrix} \eta^{\dagger\dot{\alpha}} \\ \xi_\alpha \end{pmatrix}$$ 4 components of same charge

$\rightsquigarrow \Psi \to e^{-ie\lambda}\Psi$

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$ 4x4 Dirac matrices (in 2x2 block form) (chiral repres., ∃ others)

$\hookrightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}$ Clifford algebra

- chirality

define $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \rightsquigarrow \{\gamma^5, \gamma^\mu\} = 0,\ (\gamma^5)^2 = \mathbb{1}$

two eigenspaces:

EV $= +1$, projector $P_R = \frac{1}{2}(1+\gamma^5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$: $P_R\Psi = \begin{pmatrix} \eta^\dagger \\ 0 \end{pmatrix} =: \Psi_R$

EV $= -1$, projector $P_L = \frac{1}{2}(1-\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$: $P_L\Psi = \begin{pmatrix} 0 \\ \xi \end{pmatrix} =: \Psi_L$

left-chiral field $\xi_\alpha \to$ left-handed e^- & right-handed e^+

right-chiral field $\eta^{\dagger\dot{\alpha}} \to$ right-handed e^- & left-handed e^+

- Dirac conjugation: $\bar{\Psi} := \Psi^\dagger \gamma^0 = (\xi^*_{\dot\alpha}, \eta^\alpha)$

  Why γ^0? rotations $(\Sigma^{ij})^\dagger = \Sigma^{ij}$

  boost $(\Sigma^{0i})^\dagger = -\Sigma^{0i}$

  but $(\Sigma^{\mu\nu})^\dagger = \gamma^0 \Sigma^{\mu\nu} \gamma^0 \rightsquigarrow \bar{\Psi}$ Lorentz-transforms correctly

  invariants / covariants:

  $\bar{\Psi}\Psi = \xi^\dagger_{\dot\alpha} \eta^{\dagger\dot\alpha} + \eta^\alpha \xi_\alpha$ scalar (1)

  $\bar{\Psi}\gamma^5\Psi$ pseudoscalar (1)

  $\bar{\Psi}\gamma^\mu\Psi$ vector (4)

  $\bar{\Psi}\gamma^\mu\gamma^5\Psi$ pseudovector (4)

  $\bar{\Psi}\gamma^{\mu\nu}\Psi = \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\bar{\Psi}\gamma_{\rho\lambda}\gamma^5\Psi$ (6)

  $\left.\right\}$ total: 16 objects $\Updownarrow$ 16 components $\bar{\Psi}_A \Psi_B$

  $\gamma^{\mu\nu} := \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

- full QED Lagrangian:

  $$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - m\bar{\Psi}\Psi$$

  invariant under $A_\mu \to A_\mu + \partial_\mu\lambda$ & $\Psi \to e^{-ie\lambda}\Psi$

fermion eqs. of motion:

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\psi = 0 \iff (i\not{D} - m)\psi = 0$$

with $\not{a} := \gamma^\mu a_\mu$ "Dirac equation"

is analog of KFG equation for boson fields

also double interpretation:

- classical field equation (ψ = Grassmann field)
- single-particle wave equation (ψ = wave equ., if A_μ small)

Maxwell eqs. of motion:

$$\partial_\nu F^{\nu\mu} = e\bar{\psi}\gamma^\mu\psi =: e\, j^\mu \quad \text{electrom. current}$$

$$\rightsquigarrow 0 = \partial_\mu \partial_\nu F^{\nu\mu} = e\,\partial_\mu j^\mu \quad \text{charge conservation}$$

recommend: Dirac biography by Graham Farmelo "the strangest man"

$\rightsquigarrow$ "Dirac eq. was a strike of genius"